Notes on domain theory and topology

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These notes have been written to complement the study of the very nice book of Gilles Dowek on "Proofs and Algorithms - An Introduction to Logic and Computability" [2] by Ying Jiang's group of students at the Institute of Software of the Chinese Academy of Sciences (ISCAS) that met more or less every Tuesday from May to August 2012.

It introduces the notion of directed-complete partial order (dCPO) that is the basis of domain theory [1] and topology.

1 Directed-complete partial orders

Let $\leq$ be an ordering relation on a set E.

The first chapter of Dowek's book starts by introducing the following notions:

Definition 1 (Limit, complete ordering, continuity) An element $l \in E$ is the *limit* of an *increasing* sequence $u_0 \leq u_1 \leq \ldots$ if l is the *least upper bound* (lub) of the set $\{u_0, u_1, \ldots\}$.

The ordering $\leq$ is *weakly complete* if every increasing sequence has a limit. It is *strongly complete* if every subset of E has a least upper bound.

A function $f : E \to E$ is *increasing* if for all $x, y \in E$, $x \leq y \Rightarrow f(x) \leq f(y)$.

In a weakly complete ordering $\leq$, a function $f : E \to E$ is *continuous* if it is increasing and if, for every increasing sequence $(u_i)_{i\geq 0}$, $\text{lub}\{f(u_i)\}_{i\geq 0} = f(\text{lub}\{u_i\}_{i\geq 0})$.

A set E equipped with an ordering relation $\leq$ is also called a *partially ordered set* or *poset*. In a *totally ordered set*, every two elements are comparable.

An increasing function is more usually called *monotone*, *monotonic* or *isotone* (it is a morphism in the category of posets).

An increasing sequence $u_0, u_1, \ldots$ is also called an ω-*chain* (ω is the first infinite ordinal), a *chain* being a non-empty totally ordered set. Because of this restriction to ω-chains, Dowek's notions of weak completeness and continuity are generally called ω-completeness (ωCPO) and ω-continuity.

Moreover, a strongly complete ordering is a *complete lattice*:

Definition 2 (Lattice) A *lattice* is a poset $(E, \leq)$ such that every two elements have a least upper bound (lub) and a greatest lower bound (glb). A lattice is *complete* if every subset of E has a lub and a glb.

An easy induction shows that, in a lattice, every *non-empty finite* subset of E has a lub and a glb. It is complete if every empty and infinite subset has a lub and a glb.

Lemma 3 A strongly complete ordering is a complete lattice.

Proof. The fact that every subset has a lub follows from the definition. For the glb, take $\mathrm{glb}(A) = \mathrm{lub}\{x \in E \mid x \leq A\}$. This is Proposition 1.2 in Dowek's book. ∎

Note that, if they exist, $\mathrm{lub}\,\emptyset = \mathrm{glb}E$ is the smallest element of E and $\mathrm{lub}E = \mathrm{glb}\,\emptyset$ the greatest.

A more abstract notion, not using sequences, is the one of *directed-complete partial order* (dCPO). This is the basis of *domain theory* used in programming languages semantics. We refer to [1] for more details. Domain theory has been introduced by Scott [5, 6, 8, 7].

Definition 4 (Directed-complete poset (dCPO)) A *directed set* D is a non-empty set such that every two elements has an upper bound (in D). A poset is *directed-complete* (dCPO) if every directed subset has a lub. In a dCPO, a function $f : E \to E$ is d-continuous (or Scott-continuous) if f is monotone and, for every directed subset D, $\mathrm{lub}f(D) = f(\mathrm{lub}D)$.

Note that, in the definition of d-continuity, $\mathrm{lub}f(D)$ exists since, by monotony of f, $f(D)$ is directed.

By an easy induction, one can check that a directed set is a poset having an upper bound for every *finite* subset (including the empty subset since it is non-empty).

A chain is a directed set. Hence, every dCPO is a chain-complete poset (*i.e.* every chain has a lub) and thus an ωCPO. Conversely, using the axiom of choice, every chain-complete poset is a non-empty (or *pointed*) dCPO [4].

In Dowek's book, it is proved (Proposition 1.1, "1st fixed-point theorem") that, in an ωCPO, every ω-continuous function $f : E \to E$ has a least fixpoint (lfp), *i.e.* there is x such that $f(x) = x$. A similar theorem holds in dCPOs:

Lemma 5 In a dCPO, every d-continuous function $f : E \to E$ has a lfp.

In Dowek's book, it is also proved (Proposition 1.3, "2nd fixed-point theorem") that, in a complete lattice, every monotone function $f : E \to E$ has a lfp. This theorem is due to Tarski [10]. The full Tarski's theorem in fact says that the set of all the fixpoints of f is itself a complete lattice. And it can be extended to chain-complete posets [3].

Lemma 6 In a chain-complete poset, the set of fixpoints of a monotone function $f : E \to E$ is chain-complete. In particular, f has a lfp.

In fact, this property of monotone functions is a charateristic of chain-complete posets: if every monotone map $f : E \to E$ has a least fixpoint, then E is chain-complete [3].

2 Topology

In Dowek's book and in the previous section, the notion of limit of a chain or directed set is defined as its lub, and a monotone function $f : E \to E$ is continuous if it permutes with lub or if, for all directed subset D, $f(\mathrm{lub}D) \leq \mathrm{lub}f(D)$ (since we always have $\mathrm{lub}f(D) \leq f(\mathrm{lub}D)$ by monotony of f).

However, in mathematics, there exists a more abstract notion of continuity based on the notion of *topology*. Let $\wp(E)$ be the set of all the subsets of E.

Definition 7 (Topology) A *topology* τ on a set E is a set of subsets of E ($\tau \subseteq \wp(E)$), called *opens*, such that:

- $\emptyset$ and E are opens: $\emptyset \in \tau$ and $E \in \tau$;
- τ is closed under arbitrary unions: if $X \subseteq \tau$, then $\bigcup X \in \tau$;
- τ is closed under *non-empty finite* intersections[1]: if $X \subseteq \tau$ and X is *finite*, then $\bigcap X \in \tau$.

A function $f : E \to E$ is τ*-continuous* if the inverse image of an open is an open: forall $U \in \tau$, $f^{-1}(U) \in \tau$.

Alternative formulation using families:

- τ is closed under arbitrary unions: if $(U_i)_{i\in I}$ is an arbitrary family of opens, then $\bigcup_{i\in I} U_i$ is an open;
- τ is closed under *non-empty finite* intersections: if $(U_i)_{1\leq i\leq n}$ is a *non-empty finite* family of opens (*i.e.* $n \geq 1$), then $\bigcap_{i\leq i\leq n} U_i$ is an open.

For instance, the standard notion of continuity on the real line $\mathbb{R}$ corresponds to the following topology: $U \subseteq \mathbb{R}$ is open if, for all $x \in U$, there is $\epsilon > 0$ such that $]x-\epsilon, x+\epsilon[\ \subseteq U$. Hence, $]0,1[$ is open, but $[0,1]$ is not. Closure by *finite* intersections only is important: for all $i \geq 1$, $]-\frac{1}{i}, +\frac{1}{i}[$ is open, but $\bigcap_{i\geq 1}]-\frac{1}{i}, +\frac{1}{i}[= \{0\}$ is not open.

We will see that the notion of d-continuity corresponds to the following topology:

[1] Since τ has a greatest element E, one can define $\bigcap \emptyset$ as E and say that τ is closed under *finite* intersection (empty or not).

Definition 8 (Scott topology) Let σ be the set of subsets U such that:

- U is upper-closed: if $x \in U$ and $x \leq y$, then $y \in U$;
- U is accessible by directed lubs: for all directed subset D with lub$D \in U$, we have $D \cap U \neq \emptyset$.

We first check that σ is indeed a topology:

Lemma 9 σ is a topology.

Proof.

- One can easily check that $\emptyset$ and E are upper-closed and accessible by directed lubs.
- Let (U_i) be a family of opens. Then, one can easily check that $\bigcup_{i \in I} U_i$ is upper-closed and accessible by directed lubs since each U_i do so.
- Let $(U_i)_{1 \leq i \leq n}$ be a non-empty finite family of opens. Then, one can easily check that $U = \bigcap_{1 \leq i \leq n}$ is upper-closed. Let now D be a directed set such that lub$D \in U$. Then, for every $i \in \{1, \ldots, n\}$, there is $x_i \in D \cap U_i$. Since D is directed, $x = \text{lub}\{x_1, \ldots, x_n\} \in D$. Since every U_i is upper-closed, we have $x \in \bigcap_{1 \leq i \leq n} U_i$. ■

We now prove that d-continuity is equivalent to σ-continuity:

Lemma 10 A function $f : E \to E$ is d-continuous iff it is σ-continuous.

Proof. We first prove that d-continuity implies σ-continuity. Let $U \in \sigma$. We have to prove that $f^{-1}(U) \in \sigma$.

- $f^{-1}(U)$ is upper-closed: if $x \in f^{-1}(U)$ and $x \leq y$ then $f(x) \in U$ and, by monotony of f, $f(y) \in U$, *i.e.* $y \in f^{-1}(U)$.
- $f^{-1}(U)$ is accessible by directed lubs: let D be a directed subset such that lub$D \in f^{-1}(U)$. Then, $f(\text{lub}D) \in U$. By d-continuity, $f(\text{lub}D) = \text{lub}f(D)$. Since U is open, $f(D) \cap U \neq \emptyset$. Therefore, $D \cap f^{-1}(U) \neq \emptyset$.

We now prove that σ-continuity implies d-continuity. To this end, first remark that $]a, +\infty[= \{x \in E \mid x \not\leq a\}$ is an open:

- $]a, +\infty[$ is upper-closed: if $x \not\leq a$ and $x \leq y$, then $y \not\leq a$. Otherwise, by transitivity, $x \leq a$.
- $]a, +\infty[$ is accessible by directed lubs: if D is directed and lub$D \not\leq a$, then there is $d \in D$ such that $d \not\leq a$. Otherwise, $D \leq a$ and lub$D \leq a$, which is not possible.

We now prove that f is d-continuous:

- f is monotone: let x and y such that $x \leq y$. We have to prove that $f(x) \leq f(y)$. If $f(x) \not\leq f(y)$, then $f(x) \in U =]f(y), +\infty[$ and $x \in f^{-1}(U)$. Since f is d-continuous and U is open, $f^{-1}(U)$ is open and, in particular, upper-closed. Thus, $y \in f^{-1}(U)$ and $f(y) \not\leq f(y)$, which is not possible.

- For all directed subset D, $f(\text{lub}D) \leq \text{lub}f(D)$: if $f(\text{lub}D) \not\leq \text{lub}f(D)$, then $f(\text{lub}D) \in U =]\text{lub}f(D), +\infty[$ and $\text{lub}D \in f^{-1}(U)$. Since f is d-continuous and U is open, $f^{-1}(U)$ is open and, in particular, accessible by directed lubs. Thus, there is $d \in D \cap f^{-1}(U)$. Hence, $f(d) \in f(D) \cap U$ and $f(d) \not\leq \text{lub}f(D)$, which is not possible. ■

A subset F of E is *closed* if its complement in E, $F - E$, is open. The set of closed sets has the following closure properties, dual to those of open sets:

- $\emptyset$ and E are closed sets[2];

- the set of closed sets is closed under *finite* unions: if X is a *finite* set of closed sets, then $\bigcup X$ is a closed set;

- the set of closed sets is closed under arbitrary intersections: if X is a set of closed sets, then $\bigcap X$ is a closed set.

Alternatively, using families:

- the set of closed sets is closed under *finite* unions: if $(F_i)_{1 \leq i \leq n}$ is a finite family of closed sets, then $\bigcup_{1 \leq i \leq n} F_i$ is a closed set;

- the set of closed sets is closed under arbitrary intersections: if $(F_i)_{i \in I}$ is an arbitrary family of closed sets, then $\bigcap_{i \in I} F_i$ is a closed set.

Finally, note that a function f is τ-continuous if, for all closed set F, $f^{-1}(F)$ is closed.

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