Automated certification of termination proofs

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Outline

Proof assistant kernels

$$(\text{conv}) \quad \frac{\Gamma \vdash t : T \quad T \simeq_\Gamma T'}{\Gamma \vdash t : T'}$$

- type checking
- type equivalence checking
  - $\Rightarrow$ $\simeq_\Gamma$ must be decidable

Equivalence on types/propositions

bigger is $\simeq_\Gamma$:

- bigger can be the deduction steps
  - $\Rightarrow$ closer are we to the mathematical practice
  - $\Rightarrow$ less uninteresting technical lemmas are needed
  - $\Rightarrow$ faster are developments
- smaller can be the proofs
  - $\Rightarrow$ more proofs can be type-checked
- more terms can be typable
  - $\Rightarrow$ easier are developments with dependent types
- closer are we to undecidability

Extensions of the Calculus of Constructions

$\simeq_\Gamma$ decidable				$\simeq_\Gamma$ not decidable	
β (CC)	$\beta\iota$ (CIC)	βR (CAC)	CCIC	OCC	CC+ext
Coquand Huet [80]	Paulin [89] Altenkirch [93] Werner [94]	Barbanera [90] Fernández Geuvers [93] Barthe [98] B. Jouannaud Okada [99] B. [01-05]	Jouannaud B. Strub [05]	Stehr [02]	Hofmann [95] Oury [05]

The Calculus of Constructions with extensionality

Hofmann [95] Oury [05]

$$\text{(beta)} \quad \frac{T \downarrow_\beta U}{T \simeq_\Gamma U}$$

$$\text{(ext)} \quad \frac{\Gamma \vdash p : T = U}{T \simeq_\Gamma U}$$

The Open Calculus of Constructions (OCC)

Stehr [02]

type checking	$\Gamma \vdash t : T$	
type inference	$\Gamma \vdash t \rightarrow : T$	
computational equality	$\Gamma \vdash !! t = u$	(using rewriting)
structural equality	$\Gamma \vdash \|\| t = u$	(same term representation)
provability	$\Gamma \vdash ?? T$	(using logic programming)

$$\text{(Red)} \quad \frac{\Gamma \vdash z \rightarrow : !!\forall x : V, P \Rightarrow t = u \quad \Gamma \vdash v : V \quad \Gamma \vdash ?? P_x^v}{\Gamma \vdash !! (t = u)_x^v}$$

implemented in first-order rewriting logic (Maude)

The Calculus of Algebraic Constructions (CAC)

Barbanera [90] Fernández [93] B. Jouannaud Okada [99] B. [01-05] ...

taking $\simeq_\Gamma = \downarrow_{\beta R}$ with a set R of higher-order rewrite rules includes:

- CIC: ι-reduction is an orthogonal constructor-based HORS
- functions with non-structurally smaller recursive calls
- (non-linear) matching on various arguments at the same time
- matching on defined symbols: $x + (y + z) \to (x + y) + z$
- non-free data types: $s(p(x)) \to x$ and $p(s(x)) \to x$ (integers)
- decision procedures like the `ring` tactic in Coq by applying rules with matching modulo associativity and commutativity

prototype of Coq+R using CiME (SVN branch "recriture", B. [03])

Termination of $\longrightarrow_{\beta R}$?

- when R is any terminating first-order rewrite system
  (Breazu-Tannen Okada [89] Barbanera [90] Dougherty [91])
- general schema / computability closure
  (B. Jouannaud Okada [99] B. [01-05])
- HORPO
  (Jouannaud Rubio [99] Walukiewicz [03])
- size-based termination
  (Xi [01] Abel [03-06] Barthe+ [04-06] B. [04-05] B. Riba [06])
  (prototype for CIC by Grégoire [05])

The Calculus of Congruent Constructions (CCC)

B. Jouannaud Strub [03]

$\simeq_\Gamma$ includes the equational theory generated by the *closed* equational hypotheses of Γ:

$$h_1 : x = y + 2, h_2 : f(x) = g(x-2) \vdash \mathit{refl\_equal}\ f(x) : f(x) = g(y)$$

two possible implementations:

- using rewriting since, by *Knuth-Bendix completion*, equational hypotheses can be turned into a terminating confluent system
- using an external specialized tool for congruence closure

prototype implemented in Maude (Strub [03])

The Calculus of Congruent Inductive Constructions (CCIC)

B. Jouannaud Strub [05]

$\simeq_\Gamma$ includes the equational theory generated by the *closed* equational hypotheses of Γ and *any Nelson-Oppen combination of stably infinite theories* (linear arithmetic, lists, arrays, etc.)

$\Rightarrow$ implementation using an external specialized tool
like haRVey, Ergo, Simplify, Omega, etc.

Going further ?

take into account *quantified* equational hypotheses present in Γ:

$N : Type, 0 : N, s : N \Rightarrow N, + : N \Rightarrow N \Rightarrow N,$
$+_0 : \forall x : N, x + 0 = x,$
$+_s : \forall xy : N, x + (sy) = s(x + y),$
$+_c : \forall xy : N, x + y = y + x,$
$+_a : \forall xyz : N, x + (y + z) = (x + y) + z, \ldots$
$h_1 : x = y + 2, h_2 : f(x) = g(x - 2) \vdash \mathit{refl\_equal}\ f(x) : f(x) = g(y)$

$\Rightarrow$ application in the definition of functors

$\Rightarrow$ requires *Knuth-Bendix completion*

Conclusion

replacing in kernels ι by R has many advantages but:

- how to check termination, confluence, etc. ?
  - $\Rightarrow$ use available automated termination provers like TPA, AProVE, TTT, CiME, etc.
- how to be sure that these tools are correct ?
  - $\Rightarrow$ certify their source code ?
    difficult, long, must be done every time the code is changed
- make them provide a certificate and certify the certificates !
  - $\Rightarrow$ certify termination criteria used by the tools
  - $\Rightarrow$ provide tactics checking criteria conditions
  - $\Rightarrow$ provide a program generating proofs from certificates

Outline

CoLoR: a Coq Library on Rewriting and termination

- project started in March 2004, first release in March 2005
- contributors:
  B., Coupet-Grimal, Delobel, Hinderer, Koprowski, Le Roux
- free download on `http://color.loria.fr/`
- publications: JFLA'06, WST'06, RTA'06

CoLoR content (1/2)

- Mathematical structures:
  - relations
  - semi-rings
- Data structures:
  - lists
  - vectors
  - integer polynomials with multiple variables
  - finite multisets
  - matrices
- Term structures:
  - algebraic terms with symbols of fixed arity
  - varyadic terms
  - simply typed lambda-terms

CoLoR content (2/2)

- Transformation techniques:
  - conversion from algebraic to varyadic terms
  - arguments filtering
  - rule elimination
  - dependency pairs
- Termination criteria:
  - polynomial interpretations
  - multiset ordering
  - recursive path ordering
  - higher-order recursive path ordering
  - dependency graph cycles
  - matrix interpretations

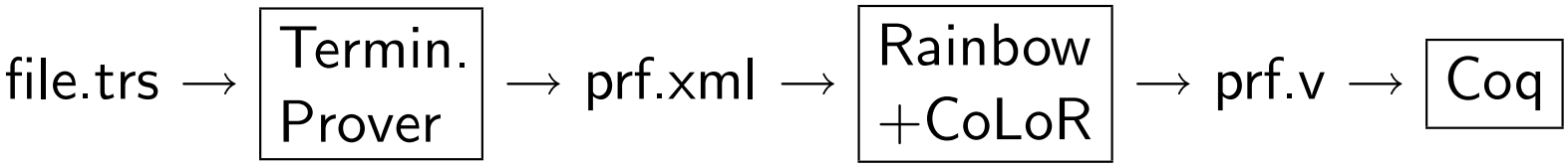
CoLoR figures

- 41,400 lines of Coq:
  - half of the size of Coq standard library
  - 5% of Coq contribs

  - Mathematical structures: 10%
  - Data structures: 29%
  - Term structures: 44%
  - Termination criteria: 17%

  - 36 inductive types or predicates (`Inductive`)
  - 115 recursive functions (`Fixpoint`)
  - 533 non-recursive definitions (`Definition`)
  - 2149 lemmas and theorems (`Lemma`)

Outline



termination proof grammar:

```
proof = MANNA_NESS red_ord | DP proof | ...
red_ord = POLY_INT poly_int | RPO prec status
          | LEX red_ord+ | ...
poly_int = symbol polynom poly_int
polynom = ...
```

XML Schema for termination proofs

```
<complexType name="ReductionOrdering">
  <choice>
    <element name="rpo" type="prf:RPO"/>
    <element name="poly_int" type="prf:PolynomialInterpretation"
    <element name="lex" type="prf:LexicographicCombination"/>
  </choice>
</complexType>

<complexType name="Proof">
  <choice>
    <element name="manna_ness" type="prf:ReductionOrdering"/>
  </choice>
</complexType>

<element name="proof" type="prf:Proof"/>
```

Example of termination problem

```
<?xml version="1.0" encoding="ISO-8859-1"?>

<problem
  xmlns="urn:rainbow.problem.format"
  xmlns:xsi="http://www.w3.org/2001/XMLSchema-instance"
  xsi:schemaLocation="urn:rainbow.problem.format problem.xsd">

<algebra>
  <signature>
    <mapping><fun>plus</fun><arity>2</arity></mapping>
    <mapping><fun>succ</fun><arity>1</arity></mapping>
    <mapping><fun>zero</fun><arity>0</arity></mapping>
  </signature>
</algebra>

<theory/>
```

```
<strategy><none/></strategy>

<rules>
  <rule> <!-- plus zero v0 = v0 -->
    <lhs>
      <app>
        <fun>plus</fun>
        <arg><app><fun>zero</fun></app></arg>
        <arg><var>0</var></arg>
      </app>
    </lhs>
    <rhs>
      <var>0</var>
    </rhs>
  </rule>
  ...
</rules>

</problem>
```

Example of termination proof

```
<?xml version="1.0" encoding="ISO-8859-1"?>

<proof
  xmlns="urn:rainbow.proof.format"
  xmlns:xsi="http://www.w3.org/2001/XMLSchema-instance"
  xsi:schemaLocation="urn:rainbow.proof.format proof.xsd">

<manna_ness>
  <poly_int>
    <mapping>
      <fun>plus</fun>
      <polynomial> <!-- 2.x_1 + 1.x_2 + 2 -->
        <monomial>
          <coef>2</coef>
          <var>1</var>
          <var>0</var>
        </monomial>
```

Coq file generated by Rainbow

```
Inductive symbol : Set :=
  | plus : symbol
  | succ : symbol
  | zero : symbol.

Lemma eq_symbol_dec : forall f g : symbol, {f=g}+{~f=g}.

Proof. decide equality. Qed.

Definition arity (s : symbol) :=
  match s with
  | plus => 2
  | succ => 1
  | zero => 0
end.
```

```
Require Import ASignature.

Definition sig := mkSignature arity eq_symbol_dec.
Definition S_plus x2 x1 := (@Fun sig plus (Vcons x2 (Vcons x1 Vn
Definition S_succ x1 := (@Fun sig succ (Vcons x1 Vnil)).
Definition S_zero := (@Fun sig zero Vnil).

Require Import ATrs.

Definition rules :=
   mkRule (S_plus S_zero (Var 0))
          (Var 0)
:: mkRule (S_plus (S_succ (Var 1)) (Var 0))
          (S_succ (S_plus (Var 1) (Var 0)))
:: nil.
```

```
Require Import Polynom.

Definition poly (f : symbol) : poly (arity f) :=
  match f as f return poly (arity f) with
    | plus => (2%Z, (Vcons 1 (Vcons 0 Vnil)))
              :: (1%Z, (Vcons 0 (Vcons 1 Vnil)))
              :: (2%Z, (Vcons 0 (Vcons 0 Vnil)))
              :: nil
    | succ => (1%Z, (Vcons 1 Vnil))
              :: (1%Z, (Vcons 0 Vnil))
              :: nil
    | zero => (1%Z, Vnil)
              :: nil
end.
```

```
Require Import MonotonePolynom.

Lemma monotony : forall f, pmonotone (poly f).

Proof.
pmonotone.
Qed.

Require Import APolyInt.

Definition PI := mkPolyInterpretation sig poly monotony.

Lemma termination : wf (red rules).

Proof.
poly_int PI.
Qed.
```

CoLoR/PolyInt/APolyInt.v

```
(* check that, for all l->r in R,  all the coefficients
of the polynomial [l]-[r]+1 are >= 0 *)

Lemma polyInterpretationTermination : forall R,
  lforall (fun lr => coef_pos (rulePoly_gt lr)) R -> WF (red R).

Proof. ... Qed.

Ltac poly_int PI :=
  match goal with
    |- WF (red ?R) =>
      apply (polyInterpretationTermination PI R);
        vm_compute; intuition; discriminate
  end.
```

Towards a certified termination competition

annual international termination competition (deadline May 21)
`http://www.lri.fr/~marche/termination-competition/`

categories: rewrite systems, logic programs, functional programs

new option this year: certified proofs !

in 2006, $3/11 = 27\%$ tools were discovered to be bugged

Current results

competition database: 1875 problems (some are open!)
in standard rewriting: 865

certified	TPA	19%	with polynomial interpretations
	+Rainbow	27%	with matrix interpretations
	+CoLoR	32%	with both
not certified	Jambox	62%	matrix interp + dependency pairs
	AProVE	73%	

TPA is developed by Adam Koprowski
`http://www.win.tue.nl/tpa/`

Related work

- A3PAT project for certifying CiME (Contejean [04])
- Isabelle/HOL termination checker (Bulwahn Krauss Nipkow [07])

Put CoLoR in your life ! :-)

`http://color.loria.fr/`

Contributions are welcome !